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Four-dimensional N=2 Field Theory and Physical Mathematics

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ABSTRACT: We give a summary of a talk delivered at the 2012 International Congress on Mathematical Physics. We review d=4, N=2 quantum field theory and some of the exact statements which can be made about it. We discuss the wall-crossing phenomenon. An interesting application is a new construction of hyperkähler metrics on certain manifolds. Then we discuss geometric constructions which lead to exact results on the BPS spectra for some d=4, N=2 field theories and on expectation values of - for example - Wilson line operators. These new constructions have interesting relations to a number of other areas of physical mathematics.

Contents

1. Introduction

The following is a brief summary of a review talk delivered at the ICMP in Aalborg, Denmark, August 2012. The powerpoint slides are available at [1]. After reviewing some standard material on d=4, N=2 quantum field theories we review some work done in a project with Davide Gaiotto and Andy Neitzke [2, 3, 4, 5, 6, 7]. A more extensive pedagogical review is in preparation and preliminary versions are available at [8]. Those notes are based on lectures recently given in Bonn, and the videos are available at [9]. Another brief summary of the construction of hyperkähler metrics is available at Andy Neitzke's homepage [10].

Let us begin with some motivation. Two important problems in mathematical physics are:

1. Given a quantum field theory (QFT), what is the spectrum of the Hamiltonian, and how do we compute forces, scattering amplitudes, operator vev's, etc?

2. Find solutions to Einstein's equations and find solutions to the Yang-Mills equations on Einstein manifolds.

The present work addresses each of these questions within the restricted context of four-dimensional QFT with N=2 supersymmetry. Regarding problem 1, in the past five years there has been much progress in finding exact results on a portion of the spectrum, the so-called "BPS spectrum," of the Hamiltonian. A corollary of this progress is that many exact results have been obtained for "line operator" and "surface operator" vacuum expectation values. Regarding problem 2, it turns out that understanding the BPS spectrum allows one to give very explicit constructions of hyperkähler metrics on certain manifolds associated to these d=4, N=2 field theories. Hyperkähler (HK) manifolds are Ricci flat, and hence are solutions to Einsteins equations. Moreover, the results on "surface operators" lead to a construction of solutions to natural generalizations of the Yang-Mills equations on HK manifolds. These are hyperholomorphic connections, defined by the condition that the curvature is of type $(1,1)$ in all complex structures. On a 4-dimensional HK manifold a hyperholomorphic connection is the same thing as a self-dual Yang-Mills instanton.

A good development in physical mathematics should open up new questions and directions of research and provide interesting links to other lines of enquiry. It turns out that solving the above problems leads to interesting relations to Hitchin systems, integrable systems, moduli spaces of flat connections on surfaces, cluster algebras, Teichmüller theory and the "higher Teichmüller theory" of Fock and Goncharov. The list goes on. There are many open problems in this field, some of which are mentioned in the conclusions.

2. d=4 N=2 field theory

The N=2 super-Poincaré algebra is a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{S} = \mathfrak{S}^0 \oplus \mathfrak{S}^1$. The even subspace is $\mathfrak{S}^0 = iso(1,3) \oplus su(2)_R \oplus \mathbb{R}^2$ where the second summand on the RHS is a global symmetry known as "R-symmetry" and the last summand is central. The odd subspace is in the representation of $\mathfrak{S}^0$ given by $\mathfrak{S}^1 = [(2,1;2) \oplus (1,2;2)]_{\mathbb{R}}$ where the last subscript is a natural reality condition. Physicists usually write the odd generators as Q_α^A and $\bar{Q}_{\dot{\alpha}}^A$ where $\alpha, \dot{\alpha}$ are spin indices and $A = 1, 2$ is an $SU(2)$ R-symmetry index. The brackets of odd generators are, using standard Bagger-Wess notation:

$$\{Q_\alpha^{\ A}, \bar{Q}_{\dot{\beta}B}\} = 2\sigma_{\alpha\dot{\beta}}^m P_m \delta_B^A$$
$$\{Q_\alpha^{\ A}, Q_\beta^{\ B}\} = 2\epsilon_{\alpha\beta}\epsilon^{AB}\bar{Z} \qquad (2.1)$$
$$\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = -2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB}Z.$$

Exact N=2 supersymmetry strongly constrains a QFT. It constrains the field content, which must be in representations of the supersymmetry algebra, and it constrains Lagrangians which, for a given field content, typically depend on far fewer parameters than in the nonsupersymmetric case. N=2 also opens up the possibility of "small" or "BPS" representations of supersymmetry, over which we have much greater analytical control.

As an example, let us consider $N = 2$ supersymmetric Yang-Mills theory (SYM) for a compact simple Lie group G. In addition to a gauge field A_μ^a, where $\mu = 0, 1, 2, 3$ and $a = 1, \ldots, \dim G$, there must be a doublet of gluinos in the adjoint representation and - very importantly - a pair of real scalar fields in the *adjoint* representation of the group.

These are usually combined into a complex scalar field with an adjoint index, φ^a. In this case the renormalizable Lagrangian is completely determined up to a choice of Yang-Mills coupling. The Hamiltonian is the sum of the standard terms and a potential energy term

$$\frac{1}{g^2} \int d^3x \text{Tr}([\varphi, \varphi^\dagger])^2. \tag{2.2}$$

This has the important consequence that there is - at least classically - a *moduli space of vacuum states*. The standard terms of the Hamiltonian set $E = B = 0$ and set φ to be a constant in space. The term (2.2) implies that in the vacuum φ must be a normal matrix so it can be diagonalized to the form $\varphi \in \mathfrak{t} \otimes \mathbb{C}$, where $\mathfrak{t}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G. Now, a standard set of arguments (due to Seiberg [11] and Seiberg and Witten [12, 13]), based on the assumption that there is no anomaly in supersymmetry and on the strong constraints that $N = 2$ supersymmetry puts on any long-distance effective action, shows that in fact, *this family of vacua is not lifted in the quantum theory.* We label the vacua as $|\Omega(u)\rangle$, with $u \in \mathcal{B} := \mathfrak{t} \otimes \mathbb{C}/W$, where W is the Weyl group. For $\mathfrak{g} = su(K)$ the quantum vacua can be characterized by the equations

$$\langle \Omega(u)|\text{Tr}\varphi^s|\Omega(u)\rangle = u_s \qquad s = 2, \ldots, K. \tag{2.3}$$

where u_s are complex numbers parametrizing the vacuum. Informally we can say

$$\langle \Omega(u)|\varphi|\Omega(u)\rangle = \text{Diag}\{a^1, \ldots, a^K\}. \tag{2.4}$$

Physical properties depend on the point $u \in \mathcal{B}$.

For generic values of $a^1, \ldots, a^K$ there is - classically - an unbroken $U(1)^r$ gauge symmetry with $r = K - 1$. The low energy theory is therefore described by an N=2 extension of Maxwell's theory, and hence we have electromagnetic fieldstrengths $F \in \Omega^2(\mathbb{R}^{1,3}; \mathfrak{t})$, and their superpartners. N=2 supersymmetry constrains the low energy effective action (LEEA) to be - roughly - of the form

$$S = \int \text{Im}\tau_{IJ}F^I * F^J + \text{Re}\tau_{IJ}F^I F^J + \text{Im}\tau_{IJ}da^I * d\bar{a}^J + \cdots \tag{2.5}$$

where $\tau_{IJ} = \frac{\theta_{IJ}}{8\pi} + \frac{4\pi i}{e_{IJ}^2}$ is a complexified coupling constant. It is a symmetric holomorphic matrix function of the vacuum parameters u. The theory contains dyonic particles with both electric and magnetic charges for the Maxwell fields. Dirac quantization shows that the electromagnetic charge γ lies in a symplectic lattice Γ, with an integral antisymmetric form: $\langle \gamma_1, \gamma_2 \rangle \in \mathbb{Z}$.

One of the key features of d=4, N=2 supersymmetry is that one can define the space of BPS states. The Hilbert space of the theory is graded by electromagnetic charge $\mathcal{H} = \oplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$. Taking the square of suitable Hermitian combinations of supersymmetry generators and using the algebra shows that in the sector $\mathcal{H}_\gamma$ there is a Bogomolnyi bound $E \geq |Z_\gamma|$ where Z_γ is the "central charge" in the N=2 supersymmetry algebra (2.1). (On the subspace $\mathcal{H}_\gamma$ the central charge operator is a γ-dependent c-number Z_γ.) The

BPS subspace of the Hilbert space is - by definition - the subspace for which the energy saturates the Bogomolnyi bound:

$$\mathcal{H}_\gamma^{BPS} := \{\psi | E\psi = |Z_\gamma|\psi\}. \tag{2.6}$$

The central charge function is linear in γ, $Z_{\gamma_1+\gamma_2} = Z_{\gamma_1} + Z_{\gamma_2}$, and is also a holomorphic function of u. It turns out that knowing $Z_\gamma(u)$ is equivalent to knowing $\tau_{IJ}(u)$.

So far, everything above follows fairly straightforwardly from general principles. But how do we actually compute $Z_\gamma(u)$ (and hence $\tau_{IJ}(u)$, and hence the low energy effective dynamics) as a function of u? In a renowned pair of papers [12, 13] Seiberg and Witten showed (for $SU(2)$ N=2 super-QCD) that $\tau(u)$ can be computed in terms of the periods of a meromorphic differential form λ on a Riemann surface Σ, both of which depend on u. They therefore showed how to determine the LEEA exactly as a function of u. They also gave cogent arguments for the exact BPS spectrum of the $SU(2)$ theory without quarks. It was therefore natural to search for the LEEA and the BPS spectrum in other d=4 N=2 theories. Extensive subsequent work showed that the Seiberg-Witten paradigm indeed generalizes to all known solutions for the LEEA of d=4 N=2 theories, namely, there is a family of Riemann surfaces Σ_u, parametrized by the moduli space of vacua, $u \in \mathcal{B}$, together with a meromorphic differential λ_u whose periods determine $Z_\gamma(u)$. The curve Σ_u and differential λ_u are called the Seiberg-Witten curve and differential, respectively. However, to this day, there is no general algorithm for computing the Seiberg-Witten curve and differential given an arbitrary d=4, N=2 field theory. It is not even clear, *a priori*, why the Seiberg-Witten paradigm should hold true for such an arbitrary theory.

One important technical detail in the Seiberg-Witten paradigm should be mentioned here. There is a complex codimension one singular locus $\mathcal{B}^{\text{sing}} \subset \mathcal{B}$ where (BPS) particles become massless. This invalidates the LEEA, which is only applicable on $\mathcal{B}^* := \mathcal{B} - \mathcal{B}^{\text{sing}}$. In terms of the Seiberg-Witten curve, some cycle pinches and a period vanishes. Related to this, the charge lattice has monodromy and hence we should speak of a local system of charge lattices over $\mathcal{B}^*$ with fiber at u denoted Γ_u.

While the LEEA of infinitely many N=2 theories was worked out in the years immediately following the Seiberg-Witten breakthrough, the BPS spectrum proved to be more difficult. It was only determined in a handful of cases, using methods which do not easily generalize to other theories [14, 15, 16, 17]. In the past five years there has been a great deal of progress in understanding the BPS spectra in an infinite number of N=2 theories. One key element of this progress has been a much-improved understanding of the "wall-crossing phenomenon" to which we turn next.

3. Wall-crossing 101

The BPS spaces defined in (2.6) are finite dimensional representations of $so(3) \oplus su(2)_R$ where $so(3)$ is the spatial rotation algebra for the little group of a massive particle. The space (2.6) clearly depends on u since $Z_\gamma(u)$ does. However, even the *dimension* of the space depends on u. As in the index theory of Atiyah and Singer, (2.6) is $\mathbb{Z}_2$-graded by

$(-1)^F$ so there is an index, in our case a kind of Witten index, which behaves much better as a function of u. It is called the second-helicity supertrace and is defined by

$$\Omega(\gamma) := -\frac{1}{2}\text{Tr}_{\mathcal{H}_\gamma^{BPS}}(2J_3)^2(-1)^{2J_3} \tag{3.1}$$

where J_3 is any generator of the rotation algebra $so(3)$. The wall-crossing phenomenon is the - perhaps surprising - fact that even the *index* can depend on u! Therefore we henceforth write $\Omega(\gamma; u)$. We hasten to add that the index is piecewise constant in connected open chambers in $\mathcal{B}$, separated by real codimension one walls. The essential physics of this "wall-crossing" is that BPS particles can form boundstates which are themselves BPS. This phenomenon was first observed in the context of two-dimensional supersymmetric field theories [18, 19], and it played an important role in the consistency of the Seiberg-Witten description of pure $SU(2)$ theory [12]. A quantitative description of four-dimensional BPS wall-crossing was first put forward in [20]. It is based on a semiclassical picture of BPS boundstates with BPS constituents. Indeed, in semiclassical analysis there is a beautiful formula due to Frederik Denef [21] which gives the boundstate radius of a boundstate of two BPS particles of charges γ_1, γ_2 in a vacuum u:

$$R_{12}(u) = \langle\gamma_1, \gamma_2\rangle \frac{|Z_{\gamma_1}(u) + Z_{\gamma_2}(u)|}{2\text{Im}Z_{\gamma_1}(u)Z_{\gamma_2}(u)^*}. \tag{3.2}$$

The Z's are functions of the moduli $u \in \mathcal{B}$. We can divide the moduli space of vacua into regions with $\langle\gamma_1, \gamma_2\rangle\text{Im}Z_{\gamma_1}(u)Z_{\gamma_2}(u)^* > 0$ and $\langle\gamma_1, \gamma_2\rangle\text{Im}Z_{\gamma_1}(u)Z_{\gamma_2}(u)^* < 0$. In the latter region the boundstate cannot exist. Now consider a path of vacua $u(t)$ which crosses a "marginal stability wall," [1] defined by

$$MS(\gamma_1, \gamma_2) := \{u|Z_{\gamma_1}(u) \parallel Z_{\gamma_2}(u) \quad \& \quad \Omega(\gamma_1; u)\Omega(\gamma_2; u) \neq 0\}. \tag{3.3}$$

As u approaches this wall through a region where the boundstate exists the boundstate radius goes to infinity. We can easily account for the states which leave the Hilbert space. They are: $\Delta\mathcal{H} = (J_{12}) \otimes \mathcal{H}_{\gamma_1}^{BPS} \otimes \mathcal{H}_{\gamma_2}^{BPS}$ where (J_{12}) is the representation of $so(3)$ of dimension $|\langle\gamma_1, \gamma_2\rangle|$. This accounts for the degrees of freedom in the electromagnetic field in the dyonic boundstate. Computing (3.1) for $\Delta\mathcal{H}$ produces the "primitive wall-crossing formula" of [20].

However, this is not the full story since when crossing $MS(\gamma_1, \gamma_2)$ other "multiparticle boundstates" of total charge $N_1\gamma_1 + N_2\gamma_2$ (where N_1, N_2 are positive integers) might also decay. The full wall-crossing formula, which describes all possible bound states which can form or decay is the "Kontsevich-Soibelman wall-crossing formula" (KSWCF) [22]. Before describing a physical derivation of that formula we first digress slightly and discuss "extended operators" or "defects" in quantum field theory, because our favorite derivation of the KSWCF uses such "line defects." We should mention, however, that there are other physical derivations of the KSWCF including [23, 24, 25]. See also the review [26].

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[1] The reason for the name is that the exact binding energy of the BPS boundstate is $|Z_{\gamma_1+\gamma_2}(u)| - |Z_{\gamma_1}(u)| - |Z_{\gamma_2}(u)|$, and hence on the wall, the states are at best marginally bound.

4. Interlude: Defects in local QFT

"Extended operators" or "defects" have been playing an increasingly important role in recent years in quantum field theory. A pseudo-definition would be that defects are local disturbances supported on positive codimension submanifolds of spacetime. For example, zero-dimensional defects are just local operators. Examples of $d = 1$ defects are familiar in gauge theory as Wilson line insertions in the Yang-Mills path integral. In four-dimensions there are interesting 't Hooft loop defects based on specifying certain singularities in the gauge field on a linking 2-sphere around a line. Recent progress has relied strongly on surface defects, where we couple a two-dimensional field theory to an ambient four-dimensional theory. These 2d4d systems play an important role below.

In general the inclusion of extended objects enriches the notion of QFT. Even in the case of topological field theory, the usual formulation of Atiyah and Segal is enhanced to "extended TQFTs" leading to beautiful relations with higher category theory. We will not need that mathematics here, but the interested reader might consult [27, 28] for further information.

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We will now use line defects to produce a physical derivation of the KSWCF. This is an argument which appears in more detail in [4, 29, 30]. We consider line defects sitting at the origin of space, stretching along the (Euclidean or Lorentzian) time direction and preserving a linear combination of supersymmetries of the form $Q + \zeta \bar{Q}$ where ζ is a phase. We generally denote such line defects by L_ζ. A good example is the supersymmetric extension of the Wilson line in N=2 SYM:

$$L_\zeta = \exp \int_{\mathbb{R}_t \times \{\vec{0}\}} \left(\frac{\varphi}{2\zeta} + A + \frac{\zeta}{2} \bar{\varphi} \right). \tag{5.1}$$

For any line defect L_ζ the Hilbert space, as a representation of the superalgebra, is modified to $\mathcal{H}_{L_\zeta}$ and in the N=2 theories it is still graded by Γ, or rather by a Γ-torsor:

$$\mathcal{H}_\gamma = \oplus_{\gamma \in \Gamma + \gamma_0} \mathcal{H}_{L_\zeta, \gamma}. \tag{5.2}$$

The physical picture of the charge sector γ is that we have effectively inserted an infinitely heavy BPS particle of charge γ at the origin of space. The *framed BPS states* are states in $\mathcal{H}_{L_\zeta, \gamma}$ which saturate a modified BPS bound. This bound applies to these modified Hilbert spaces and is $E \geq -\mathrm{Re}(Z_\gamma/\zeta)$. Once again we can define a *framed BPS index*:

$$\overline{\Omega}(L_\zeta; \gamma) := \mathrm{Tr}_{\mathcal{H}_{L_\zeta, \gamma}^{BPS}} (-1)^{2J_3}. \tag{5.3}$$

If we consider line defects of type L_ζ then these framed BPS indices will be piecewise constant in ζ and u but again exhibit wall-crossing, this time across "BPS walls" defined by

$$W_\gamma := \{(u, \zeta) | Z_\gamma(u)/\zeta < 0 \qquad \& \qquad \Omega(\gamma; u) \neq 0\}. \tag{5.4}$$

The physical significance of these walls is that when (u, ζ) are close to the wall there is a subsector of $\mathcal{H}_{L_\zeta}^{BPS}$ which is described -semiclassically - by states in which a collection of BPS particles with charges of the form $n\gamma$, with $n > 0$, is bound to a defect in charge sector γ_c to make a framed BPS state with boundstate radius

$$r_\gamma = \frac{\langle \gamma, \gamma_c \rangle}{2\mathrm{Im}Z_\gamma(u)/\zeta}. \tag{5.5}$$

In fact since BPS particles of charge $n\gamma$ for $n > 0$ can bind in arbitrary numbers to the core defect, (this is possible since they feel no relative force) there is an entire Fock space of boundstates of these so-called "halo particles." When crossing the wall this entire Fock space appears or disappears in the framed Hilbert space. Exactly the same physical picture underlies the "semi-primitive wall-crossing formula" of [20].

An elegant way to express this wall-crossing mathematically is the following. Introduce the framed BPS degeneracy generating function

$$F(L) := \sum_\gamma \overline{\underline{\Omega}}(L; \gamma) X_\gamma \tag{5.6}$$

where $X_{\gamma_1} X_{\gamma_2} = (-1)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1 + \gamma_2}$ generate the twisted algebra of functions on an algebraic torus $\Gamma^* \otimes \mathbb{C}^*$. When crossing a BPS wall W_γ the charge sectors of the form $\gamma_c + N\gamma$ gain or lose a Fock space factor:

$$X_{\gamma_c} \to (1 - (-1)^{\langle \gamma_1, \gamma_2 \rangle} X_\gamma)^{\langle \gamma, \gamma_c \rangle \Omega(\gamma)} X_{\gamma_c} \tag{5.7}$$

Once again the factor $\langle \gamma, \gamma_c \rangle$ accounts for degrees of freedom in the electromagnetic field. Since the wall-crossing factor depends on γ_c, the change of $F(L)$ across a BPS wall W_γ is given by the action of a *differential operator*: $F(L) \to K_\gamma^{\Omega(\gamma)} F(L)$ where

$$K_\gamma = (1 - (-1)^{D_\gamma} X_\gamma)^{D_\gamma} \tag{5.8}$$

and $D_\gamma X_\rho = \langle \gamma, \rho \rangle X_\rho$. We now consider a point u_* on the marginal stability wall $MS(\gamma_1, \gamma_2)$. The intersection of the BPS walls $W_{r_1\gamma_1 + r_2\gamma_2}$ which go through u_* and have $r_1 r_2 \geq 0$ defines a complex codimension one locus in $\mathcal{B}^*$. Now consider two small paths linking this locus with one path in a region where $\mathrm{Im}Z_1\bar{Z}_2 > 0$ and the other in the region $\mathrm{Im}Z_1\bar{Z}_2 < 0$. On the one hand the generating function $F(L)$ is is well-defined at the endpoints of the paths: There is no monodromy in the continuous evolution of $F(L)$ around the loop. On the other hand transport of $F(L)$ along the path leads to a sequence of transformations by $K_\gamma^{\Omega(\gamma)}$ each time the point $u(\tau)$ on the path goes through a wall W_γ. These two statements together [2] imply the KSWCF:

$$\prod_{\nearrow} K_{r_1\gamma_1 + r_2\gamma_2}^{\Omega(r_1\gamma_1 + r_2\gamma_2; -)} = \prod_{\searrow} K_{r_1\gamma_1 + r_2\gamma_2}^{\Omega(r_1\gamma_1 + r_2\gamma_2; +)} \tag{5.9}$$

where the product is over pairs of nonnegative integers (r_1, r_2). The product with $\nearrow$ is ordered with r_1/r_2 increasing from left to right, while that with $\searrow$ is ordered with r_1/r_2

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[2]plus an important detail that there be "sufficiently many" line defects

decreasing from left to right. The $\pm$ in Ω refers to the BPS degeneracies on either side of the wall. Knowing the $\Omega(r_1\gamma_1 + r_2\gamma_2; -)$ we compute the LHS of the equation. Given an ordering of the K_γ factors there is a unique factorization of this product of the form in (5.9). Hence, given $\Omega(r_1\gamma_1 + r_2\gamma_2; -)$ the $\Omega(r_1\gamma_1 + r_2\gamma_2; +)$ are uniquely determined. Equation (5.9) is therefore a wall-crossing formula.

Two examples serve to illustrate the theory well. If $\Gamma = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$ and $\langle \gamma_1, \gamma_2 \rangle = +1$ then

$$K_{\gamma_1} K_{\gamma_1} = K_{\gamma_1} K_{\gamma_1+\gamma_2} K_{\gamma_2}. \tag{5.10}$$

This identity is easily verified. It is related to consistency of simple superconformal field theories ("Argyres-Douglas theories") as well as to coherence theorems in category theory, 5-term dilogarithm identities, and a number of other things. Our second example again takes $\Gamma = \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$ but now with $\langle \gamma_1, \gamma_2 \rangle = +2$. Then

$$K_{\gamma_2} K_{\gamma_1} = \Pi_L K_{\gamma_1+\gamma_2}^{-2} \Pi_R \tag{5.11}$$

$$\begin{aligned} \Pi_L &= \prod_{n=0\nearrow\infty} K_{(n+1)\gamma_1+n\gamma_2} = K_{\gamma_1} K_{2\gamma_1+\gamma_2} \cdots \\ \Pi_R &= \prod_{n=\infty\searrow0} K_{n\gamma_1+(n+1)\gamma_2} = \cdots K_{\gamma_1+2\gamma_2} K_{\gamma_2} \end{aligned} \tag{5.12}$$

This identity perfectly captures the wall-crossing of the BPS spectrum found in the original example of Seiberg and Witten [12], a remark due to Frederik Denef. The corresponding identities for the cases $\langle \gamma_1, \gamma_2 \rangle \neq 0, 1, 2$ are considerably wilder.

We stress that this is only half the battle. The wall-crossing formula only describes the *change* of the BPS spectrum across a wall of marginal stability. It does *not* determine the BPS spectrum! For a certain (infinite) class of N=2 theories - the theories of class S - we can do better and give an algorithm to determine the BPS spectrum, as we describe below.

6. Reduction to three dimensions and hyperkähler geometry

Interesting relations to hyperkähler geometry emerge when we compactify N=2 theories on a circle of radius R. At energy scales much lower than $1/R$ the theory is described by a supersymmetric sigma model with target space $\mathcal{M}$ which comes with a natural torus fibration over $\mathcal{B}$ [31]. The presence of 8 supersymmetries means that $\mathcal{M}$ must carry a hyperkähler metric. In the large R limit this metric can be easily solved for, but at finite values of R there are nontrivial quantum corrections. The idea of the construction of [2] is to find a suitable set of functions on the twistor space of $\mathcal{M}$ from which one can construct the metric. The required functions turn out to be solutions to an explicit integral equation closely resembling Zamolodchikov's thermodynamic Bethe ansatz.

The low energy three-dimensional sigma model has scalar fields $a^I(x) \in \mathcal{B}$ descending from the scalars in four dimensions, as well as two periodic scalars $\theta_e^I(x)$ and $\theta_{m,I}(x)$ for each dimension $I = 1, \ldots, r$ of $\mathfrak{t}$. We can think of $\theta_e^I(x) = \oint_{S^1} A$ as the "Wilson loop scalar" and $\theta_{m,I}(x)$ as an electromagnetic dual scalar, coming from dualization of the

three-dimensional gauge field. This leads to a picture of the target space as a fibration by tori, whose generic fiber is $\Gamma_u^* \otimes \mathbb{R}/2\pi\mathbb{Z}$. In this way we find a direct relation to integrable systems. The semiflat metric on this space is computed in a straightforward way from the reduction of the four-dimensional LEEA of Seiberg-Witten and leads to

$$g^{\text{sf}} = da^I R \text{Im}\tau_{IJ} d\bar{a}^J + \frac{1}{R} dz_I (\text{Im}\tau)^{-1,IJ} d\bar{z}_J \tag{6.1}$$

where $dz_I = d\theta_{m,I} - \tau_{IJ} d\theta_e^J$. This metric will receive quantum corrections.

The best way to approach the quantum corrections is to form the twistor space $Z := \mathcal{M} \times \mathbb{C}P^1$ which comes with a fibration $p : Z \to \mathbb{C}P^1$. A theorem of Hitchin says that putting a hyperkähler metric on $\mathcal{M}$ is equivalent to putting holomorphic data on Z so that the fiber $p^{-1}(\zeta)$ above a point $\zeta \in \mathbb{C}P^1$ is $\mathcal{M}$ in complex structure ζ. Moreover there is a holomorphic 2-form form $\varpi \in \Omega_{Z/\mathbb{C}P^1}^2 \otimes \mathcal{O}(2)$ which restricts on each fiber to the holomorphic symplectic form ϖ_ζ of $\mathcal{M}^\zeta$ and which, as a function of ζ, has a three-term Laurent expansion in $\zeta \in \mathbb{C}^*$:

$$\varpi_\zeta = \zeta^{-1}\omega_+ + \omega_3 + \zeta\omega_-. \tag{6.2}$$

Here ω_+ is a holomorphic $(2,0)$ form in complex structure $\zeta = 0$ and ω_3 is the Kähler form of the metric.

The strategy of the construction is to find ϖ_ζ by covering $\mathcal{M}$ with coordinate charts of the form

$$\mathcal{U} = \Gamma^* \otimes \mathbb{C}^* \cong \underbrace{\mathbb{C}^* \times \cdots \mathbb{C}^*}_{2r}. \tag{6.3}$$

The algebraic torus has a canonical set of "Darboux functions" Y_γ given (up to a sign) [3] by evaluation with $\gamma \in \Gamma$ and satisfying $Y_{\gamma_1}Y_{\gamma_2} = (-1)^{\langle\gamma_1,\gamma_2\rangle}Y_{\gamma_1+\gamma_2}$. In terms of these we can write a canonical holomorphic symplectic form ϖ_T by choosing a basis $\{\gamma_i\}$ for Γ and writing $\varpi_T = C^{ij} d\log Y_{\gamma_i} \wedge d\log Y_{\gamma_j}$ where C_{ij} is the symplectic form of Γ in that basis. Thus, we seek suitable holomorphic maps

$$\mathcal{Y} : \mathcal{U} \times \mathbb{C}^* \to \Gamma^* \otimes \mathbb{C}^*, \tag{6.4}$$

where the second factor in the domain is the twistor sphere stereographically projected, such that $\varpi_\zeta = \mathcal{Y}^*(\varpi_T)$ has a 3-term Laurent expansion.

For the semiflat metric one can solve for these "Darboux functions" in a straightforward way to obtain

$$\mathcal{Y}_\gamma^{\text{sf}} = \exp\left[\pi R\zeta^{-1}Z_\gamma + i\theta_\gamma + \pi R\zeta\bar{Z}_\gamma\right] \tag{6.5}$$

where θ_γ is a linear combination of θ_e^I and $\theta_{m,I}$ such that $\mathcal{Y}_\gamma^{\text{sf}}\mathcal{Y}_{\gamma'}^{\text{sf}} = (-1)^{\langle\gamma,\gamma'\rangle}\mathcal{Y}_{\gamma+\gamma'}^{\text{sf}}$. The goal, then, is to find the quantum corrections: $\mathcal{Y}_\gamma = \mathcal{Y}_\gamma^{\text{sf}}\mathcal{Y}_\gamma^{\text{quant.corr.}}$. The desired properties of the exact functions $\mathcal{Y}_\gamma(u, \theta_e, \theta_m; \zeta) = \mathcal{Y}^*(Y_\gamma)$ leads to a list of conditions which are

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[3]The sign is determined by a mod-two quadratic refinement of the intersection form.

equivalent to a Riemann-Hilbert problem in the complex ζ-plane. This RH problem is then solved by the integral equation

$$\log \mathcal{Y}_\gamma = \log \mathcal{Y}_\gamma^{\mathrm{sf}} + \frac{1}{4\pi i} \sum_{\gamma_1 \in \Gamma} \Omega(\gamma_1; u) \langle \gamma_1, \gamma \rangle \int_{\ell_{\gamma_1}} \frac{d\zeta_1}{\zeta_1} \frac{\zeta_1 + \zeta}{\zeta_1 - \zeta} \log(1 - \mathcal{Y}_{\gamma_1}(\zeta_1)), \qquad (6.6)$$

where ℓ_γ is the projection, at fixed u, of W_γ to $\mathbb{C}^*$. This equation can be solved by iteration for sufficiently large R and for sufficiently tame BPS spectrum. (We expect a typical field theory to be "tame," but a typical black hole spectrum will definitely not be tame. New ideas are needed to apply these techniques to supergravity. See [32] for the state of the art.) The $\mathcal{Y}_\gamma(\zeta)$ jump discontinuously across the BPS walls in the ζ plane, but that discontinuity is a symplectic transformation, so that ϖ_ζ is continuous. Note well that the BPS spectrum is an important input into (6.6). As we have explained, it is discontinuous in u because of wall-crossing. Nevertheless, across walls of marginal stability in $\mathcal{B}$ the metric is continuous, thanks to the KSWCF. Indeed, one can reverse the logic: Physically no discontinuity in the metric is expected across marginal stability walls, and therefore we *derive* the KSWCF [2].

The "Darboux functions" $\mathcal{Y}_\gamma$ have other useful applications. For example, they can be used to write exact results for expectation values for line defects. For example, wrapping a line defect of type ζ around the compactification circle produces a *local operator* $\mathrm{tr}L_\zeta$ in the three-dimensional sigma model. The vacua of the model are points $m \in \mathcal{M}$. In [4] it is argued that the vev of this operator in the vacuum m is

$$\langle \mathrm{tr}L_\zeta \rangle_m = \sum_\gamma \overline{\underline{\Omega}}(L_\zeta; \gamma) \mathcal{Y}_\gamma(m, \zeta). \qquad (6.7)$$

A related formula leads to a natural deformation quantization of the algebra of holomorphic functions on $\mathcal{M}^\zeta$. An extension of the above integral equations leads to a construction of hyperholomorphic connections on $\mathcal{M}$ [5].

7. Theories of class S

We now turn to a rich set of examples of d=4, N=2 theories, known as the "theories of class S." The "S" is for "six" because these are N=2 theories which descend from six-dimensional theories. In these theories many physical quantities have elegant descriptions in terms of Riemann surfaces and flat connections.

The construction is based on an important claim arising from string theory, namely, that there is a family of stable interacting UV-complete field theories with six-dimensional (2,0) superconformal symmetry [33, 34, 35]. These theories have not yet been constructed - even by physical standards - but some characteristic properties of these hypothetical theories can be deduced from their relation to string theory and M-theory. For a review based on this philosophy see [4], §7.1 or the preliminary notes [8] .

In order to construct theories of class S we begin with such a nonabelian (2,0) theory in six dimensions, $S[\mathfrak{g}]$, where $\mathfrak{g}$ is a simple and simply laced compact real Lie algebra. The theory has half-BPS codimension two defects D. We compactify the theory on a Riemann

surface C, referred to as the "ultraviolet curve." The surface C has marked points $\mathfrak{s}_n$ and we put defects D_n at $\mathfrak{s}_n$. Then we partially topologically twist, by embedding $so(2)$ into the $so(5)_R$ R-symmetry of the $(2,0)$ superconformal algebra and identifying with the algebra of the structure group of the tangent bundle TC. The resulting theory - at least formally - only depends on the conformal class of the metric through the overall area. In the limit where the area of C shrinks to zero[4] we obtain a four-dimensional quantum field theory denoted $S(\mathfrak{g}, C, D)$. This construction goes back to [37]. It has a dual version given by geometric engineering in [38]. The topological twisting, defects, and relations to Hitchin systems were given in [3]. The construction was then further developed in a brilliant paper of Gaiotto [39].

Although it will not play any direct role in the rest of our story, we must digress briefly to comment on one important insight from [39] which we regard as very deep. Defects have global symmetries. The theory $S(\mathfrak{g}, C, D)$ has a global symmetry group which includes a product over n of the global symmetries of D_n. For suitable defects D_n (known as "full defects") the global symmetry is just a compact group G with Lie algebra $\mathfrak{g}$. Therefore, if we have two Riemann surfaces C_L and C_R with collections of defects D_L and D_R containing at least one such full defect in each collection we can consider the global symmetry factor G from each surface and gauge it with parameter τ. This produces a new four-dimensional theory $S(\mathfrak{g}, C_L, D_L) \times_{G,\tau} S(\mathfrak{g}, C_R, D_R)$. On the other hand, given marked points $\mathfrak{s}_L$ and $\mathfrak{s}_R$ on C_L and C_R, respectively we can choose local coordinates z_L and z_R and form a new glued Riemann surface $C_L \times_q C_R$ by identifying $z_L z_R = q$. Therefore we can form a new quantum field theory $S(\mathfrak{g}, C_L \times_q C_R, D_{LR})$, where D_{LR} is the union of the sets of left and right defects, omitting the two associated with the glued marked points. Gaiotto's conjecture is that these two four-dimensional N=2 theories are in fact the same, provided we identify $q = e^{2\pi i \tau}$. Many beautiful results flow from this observation. It is probably the fundamental reason for the AGT conjecture [40], although that intuition has not yet been made very precise. One precise mathematical version of this phenomenon, related to Higgs branches of vacua of these theories, is described in [41].

Most "natural" d=4, N=2 theories are of class S. For example, the N=2 extension of $SU(K)$ Yang-Mills coupled to quark flavors in the fundamental representation is of class S. Moreover, there are infinitely many theories of class S with no known Lagrangian description such as the Argyres-Douglas theories described in [3] or the higher rank superconformal fixed points associated with three-punctured spheres ("trinion theories") which were discovered in [39].

One of the nicest properties of these theories is their close relation to Hitchin systems. This can be seen very directly [3] by considering the compactification of the $(2,0)$ theory on $S^1 \times C$. Compactifying in either order, and using the crucial fact that the long distance dynamics of the $(2,0)$ theory on a circle of radius R is described by nonabelian five-dimensional SYM with $g_{YM}^2 \sim R$, shows that for these theories $\mathcal{M}$ can be identified with the moduli space of solutions to Hitchin's equations for a gauge connection and "Higgs

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[4]There can be subtleties in taking this limit if there are too few or nongeneric defects [36].

field" on C:

$$F + R^2[\varphi, \bar{\varphi}] = 0, \tag{7.1}$$

$$\bar{\partial}_A \varphi := d\bar{z} \left(\partial_{\bar{z}} \varphi + [A_{\bar{z}}, \varphi] \right) = 0, \tag{7.2}$$

$$\partial_A \bar{\varphi} := dz \left(\partial_z \bar{\varphi} + [A_z, \bar{\varphi}] \right) = 0. \tag{7.3}$$

Here A is a unitary connection on an Hermitian vector bundle over C, φ is an adjoint valued $(1,0)$-form field and $\bar{\varphi} = \varphi^\dagger$ is its Hermitian conjugate. A defect D_n at $\mathfrak{s}_n$ induces a singularity in the Higgs field of the form

$$\varphi \sim \frac{\mathfrak{r}_n}{z^{\ell_n}} dz + \cdots \qquad \ell_n \geq 1 \tag{7.4}$$

where z is a local coordinate near $\mathfrak{s}_n$ and ℓ_n and $\mathfrak{r}_n$ depend on D_n. The physics depends on ℓ_n and $\mathfrak{r}_n$ in a way which is still being understood. The state of the art is summarized nicely in [42]. Later we will use an important connection to complex flat connections: If (φ, A) solve the Hitchin equations and $\zeta \in \mathbb{C}^*$ then

$$\mathcal{A}(\zeta) := \frac{R}{\zeta} \varphi + A + R\zeta\bar{\varphi} \tag{7.5}$$

is flat: $d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0$. Conversely given a family of such complex flat connections, $\mathcal{A}(\zeta)$ with a three-term Laurent expansion, (φ, A) solve the Hitchin equations.

We now state how the Seiberg-Witten curve and differential, the charge lattice, the Coulomb branch, the BPS states, and a natural class of line and surface defects can all be formulated geometrically in terms of the geometry and topology of the UV curve C and its associated flat connection $\mathcal{A}$.

First, the Seiberg-Witten curve is simply

$$\Sigma := \{\lambda | \det(\lambda - \varphi) = 0\} \subset T^*C \tag{7.6}$$

and it inherits a canonical differential λ which serves as the Seiberg-Witten differential. For $\mathfrak{g} = su(K)$, the map $\pi : \Sigma \to C$ is a K-fold branched cover and this equation can be written as

$$\lambda^K + \lambda^{K-2}\phi_2 + \cdots + \phi_K = 0 \tag{7.7}$$

where ϕ_j are meromorphic j-differentials with prescribed singularities at $\mathfrak{s}_n$. From this we deduce that $\mathcal{B} := \{u = (\phi_2, \ldots, \phi_K)\}$ is a torsor for a space of meromorphic differentials on C. Similarly the local system of charges is[5] $\Gamma = H_1(\Sigma; \mathbb{Z})$.

The geometric formulation of BPS states in these theories goes back to [38, 43, 44]. We take $\mathfrak{g} = su(K)$. We label the sheets of the covering $\pi : \Sigma \to C$ by $i, j = 1, \ldots, K$. We define a *WKB path of phase* ϑ to be a local solution of a differential equation on C:

$$\langle \lambda_i - \lambda_j, \partial_t \rangle = e^{i\vartheta} \tag{7.8}$$

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[5]Actually, Γ is a subquotient. We will ignore this subtlety in this brief review for simplicity.

where i, j is an ordered pair of sheets of the covering. [6] The marked points $\mathfrak{s}_n$ act like attractors for the WKB paths. Therefore, for generic initial point and generic ϑ both ends of a WKB path tend to such marked points. One interesting exception is a WKB path beginning on a branchpoint. But once again, for a generic ϑ, the other end of such a WKB path terminates on a marked point. The adjective *generic* used above is quite important. For special values of ϑ we can have *string webs*. These are closed WKB paths, or connected graphs with all endpoints (if any) on branch points. The graphs comprising string webs are allowed to have trivalent vertices, known as *string junctions*. The three legs of the string junction consist of ingoing ij and jk WKB paths with an outgoing ik WKB path.

There is a geometrical construction, beginning with the six-dimensional $(2, 0)$ theory and any closed continuous path $\wp \subset C$, which produces a line defect in $S(\mathfrak{g}, C, D)$. The construction also depends on an angle ϑ so we denote these line defects as $L_{\wp, \vartheta}$. The "Darboux expansion" (6.7) together with a relation of $\mathcal{Y}_\gamma$ to Fock-Goncharov coordinates on moduli spaces of flat connections allows us to write physically interesting exact results for expectation values of such line defects. For example, for N=2 SU(2) SYM the vev of the Wilson line operator (5.1) wrapped around a Euclidean time circle of radius R is, exactly,

$$\langle \mathrm{tr} L_\zeta \rangle = \sqrt{\mathcal{Y}_{\gamma_e}} + \frac{1}{\sqrt{\mathcal{Y}_{\gamma_e}}} + \sqrt{\mathcal{Y}_{\gamma_e + \gamma_m}}. \tag{7.9}$$

The first two terms, with $\mathcal{Y} \to \mathcal{Y}^{\mathrm{sf}}$ give the naive semiclassical approximation. The third term is exponentially small. This, together with the the full sum of instanton corrections to $\mathcal{Y}^{\mathrm{sf}}$ give the complete set of the quantum corrections. It is not an accident that this expression bears a very strong relation to the expectation value of a length operator in quantum Teichmüller theory [47].

There is one last construction for theories of class S we will need [45, 46, 5]. This is the canonical surface defect $\mathbb{S}_z$ associated with any point $z \in C$. It is a 1+1 dimensional QFT located at, say, $x^1 = x^2 = 0$ in four-dimensions and coupled to the ambient four-dimensional theory $S(\mathfrak{g}, C, D)$. The main fact we need about this theory is that (so long as z is not a branch point of $\pi : \Sigma \to C$) it has massive vacua in 1-1 correspondence with the preimages $z^{(i)} \in \Sigma$ of z under π. Moreover, in the theory $\mathbb{S}_z$ there are solitons interpolating between vacua $z^{(i)}$ and $z^{(j)}$ for $i \neq j$. These two-dimensional solitons are represented geometrically by *open string webs* which are defined as above for string webs but one end of the graph must end at z.

8. Spectral Networks

As we have emphasized, the KSWCF by itself does not give us the BPS spectrum. For theories of class S we can solve this problem, at least in principle, with the technique of spectral networks [6]. Spectral networks are combinatorial objects associated to a branched covering of Riemann surfaces $\pi : \Sigma \to C$. They are networks $\mathcal{W}_\vartheta \subset C$ defined by the physics of two-dimensional solitons on the surface defect $\mathbb{S}_z$. Segments in the network

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[6]For $su(2)$ a WKB path is just the trajectory of a quadratic differential ϕ_2. These have been widely studied in the mathematical literature. We think the generalization to $K > 2$ is very rich and interesting.

are constructed from WKB paths of phase ϑ according to local rules given in [6]. There can be interesting discontinuous changes in $\mathcal{W}_\vartheta$ as ϑ is varied. Some amusing movies of these morphisms of spectral networks can be viewed at A. Neitzke's homepage [48]. The essential jumps of the spectral networks happen precisely at those values of ϑ which are the phases of central charges of four-dimensional BPS states. Indeed, one can write very explicit formulae for the BPS degeneracies $\Omega(\gamma; u)$ in the theories $S(su(K), C, D)$ in terms of the combinatorics of the change of the spectral network $\mathcal{W}_\vartheta$ as ϑ passes through such a critical value [6]. Spectral networks have at least three nice applications to mathematics.

The first application comes from specializing the construction of the hyperholomorphic connections mentioned above to the theories of class S. The extra integral equations in this case are generalizations of the Gelfand-Levitan-Marchenko equation of integrable systems theory and give in principle a way to construct explicit solutions to Hitchin's equations on C [5].

A second, closely related, application is that they provide the essential data needed to construct a holomorphic symplectic "nonabelianization map"

$$\Psi_\mathcal{W} : \mathcal{M}(\Sigma, GL(1); \mathfrak{m}) \to \mathcal{M}_F(C, GL(K), \mathfrak{m}) \tag{8.1}$$

which maps flat $GL(1, \mathbb{C})$ connections on Σ with specified monodromy $\mathfrak{m}_n^{(i)}$ around the lifts $\mathfrak{s}_n^{(i)}$ to flat $GL(K, \mathbb{C})$ connections on C with specified conjugacy classes of monodromy and flag structure at $\mathfrak{s}_n$. The map depends on a choice $\mathcal{W}$ of spectral network. The holonomies of the flat connection ∇^{ab} such that $\Psi_\mathcal{W}(\nabla^{\mathrm{ab}}) = \nabla$ define a set of holomorphic functions $\mathcal{Y}_\gamma = \exp \oint_\gamma \nabla^{\mathrm{ab}}$ in a chart $\mathcal{U}_\mathcal{W} \subset \mathcal{M}_F(C, GL(K), \mathfrak{m})$ where $\Psi_\mathcal{W}$ is invertible. Choosing a basis for Γ we then obtain a local coordinate system in the chart $\mathcal{U}_\mathcal{W}$. These coordinates depend on the spectral network. Comparing the coordinates across two charts, where $\mathcal{W}$ and $\mathcal{W}'$ are related by a simple morphism associated with a four-dimensional BPS state, leads to a change of coordinates closely resembling a cluster transformation. The coordinates $\mathcal{Y}_\gamma$ thereby provide a system of coordinates on moduli spaces of flat connections which appear to generalize the cluster coordinates of Thurston, Penner, Fock, and Fock and Goncharov. For the case $K = 2$, and in some nontrivial examples with $K > 2$, they coincide with coordinates defined by Fock and Goncharov, as shown in [3, 7], respectively.

The third application is to WKB theory. The $K \times K$ matrix equation on C:

$$\left(\frac{d}{dz} + \mathcal{A}\right)\psi = 0 \tag{8.2}$$

is an ODE generalizing the Schrodinger equation (which occurs with $K = 2$). If $\mathcal{A}$ is of the form (7.5) then we can study the $\zeta \to 0$ (or $\zeta \to \infty$) asymptotics at fixed (φ, A). The extension from $K = 2$ to $K > 2$ is nontrivial. The spectral networks can be interpreted as the Stokes lines for this problem [6].

9. Conclusions

In conclusion, we have a good physical understanding of wall-crossing, and some improved understanding of how to compute the BPS spectrum, at least for theories of class S. Com-

pactification on a circle leads to a new construction of hyperkähler metrics and hyperholomorphic connections. As a by-product we find many new and nontrivial results on line and surface defects and their associated BPS spectra, again in theories of class S.

Among the many open problems and future directions in this field we mention but a few. One problem is to make the spectral network technique more effective. Another is to give a direct relation to other recent works which have made important progress in the computation of the BPS spectra of N=2 theories, e.g. through BPS quivers [49, 50, 51], or geometric engineering [52, 53]. One natural question is whether it is possible to classify d=4, N=2 theories, and whether the theories of class S constitute - in some sense - "most" N=2 theories. Another interesting problem is whether the construction of hyperkähler metrics described above can be used to produce explicit metrics on - say - K3 surfaces. In another direction, the independence of the twisted theory from the Kähler class of the metric on C, together with the Gaiotto gluing conjecture mentioned above implies that, in some sense, $(2,0)$ theories can be used to define a notion of "two-dimensional conformal field theories valued in four-dimensional theories." It would be interesting to make that sense mathematically precise.

Finally, there are three broader points we would like to stress. First: Seiberg and Witten's breakthrough in 1994 opened up many interesting problems. Some were quickly solved, but some, related to the computation of the BPS spectrum, remained stubbornly open. The past five years has witnessed a renaissance of the subject, with a much deeper understanding of the BPS spectrum and of the line and surface defects in these theories. Second: This progress has involved nontrivial and surprising connections to other aspects of physical mathematics including hyperkähler geometry, cluster algebras, moduli spaces of flat connections, Hitchin systems, integrable systems, Teichmüller theory,..., the list goes on. Third, and perhaps most importantly, we have seen that the mere existence of the six-dimensional $(2,0)$ theories leads to a host of nontrivial results in quantum field theory. Indeed, in this brief review we have not mentioned a large body of parallel beautiful and nontrivial work on d=4 N=2 theories which has been done over the past few years by many physicists. All this progress sharply intensifies the urgency of the open problem of formulating 6-dimensional superconformal theories in a mathematically precise way. Many physicists regard this as one of the most outstanding problems in physical mathematics.

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